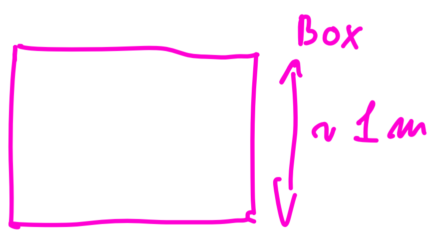
2.2) The Boltzmann Equation

2.2.1) The relevant time scales

$$\boxed{\underbrace{\partial_t f_1(\vec{q}_1,\vec{p}_1,t) + \{f_1, H\}}_{\text{free evolution}} = \underbrace{\int d\vec{q}_2\, d\vec{p}_2\, \frac{\partial V(\vec{q}_1-\vec{q}_2)}{\partial \vec{q}_1}\cdot\frac{\partial f_2(\vec{q}_1,\vec{p}_1,\vec{q}_2,\vec{p}_2)}{\partial \vec{p}_1}}_{\text{interaction term}}} \qquad (F_1)$$

Comment: $V=0 \Rightarrow \partial_t f_1 + \{f_1, H\} = 0$ is a closed equation.
"Collision free Boltzmann equation" studied in Part 2.
$\Rightarrow$ The difficulty is to deal with the right-hand side.

Involved time-scales:



All time and length scales of $V(\vec{q})$ are macroscopic.

The solution of $\partial_t f = -\frac{1}{\tau} f$ is $f(t) = f(0)\, e^{-\frac{t}{\tau}}$ $\Rightarrow$ τ is a relaxation time

$[\{f_1, H\}] = \left[\frac{f_1}{\tau_F}\right] \Rightarrow \tau_F$ is the relaxation time scale of the gas without interactions.

$\tau_F \sim$ time to cross the box $\simeq \frac{L}{v} \Rightarrow v = ?$

$m_{N_2} = \frac{2\times 14\times 10^{-3}}{N_A}$; $\frac{1}{2} m_{N_2} \langle \vec{v}^{\,2}\rangle = \frac{3}{2} k_B T \Rightarrow v = \sqrt{\langle \vec{v}^{\,2}\rangle} \simeq \sqrt{\frac{3 k_B T}{m_{N_2}}}$

$\Rightarrow v \simeq 500\,m/s \Rightarrow \tau_F \simeq 10^{-2}\,s \Rightarrow$ time scale over which $\{f_1, H_1\}$ makes f_1 relax

$\left[\int d\vec{q}_2\, d\vec{p}_2 \, \frac{\partial f_2}{\partial \vec{p}_1} \cdot \frac{\partial V}{\partial \vec{q}_1} \right] = \left[\frac{f_1}{\tau_{int}} \right]$; τ_{int} the time it takes for interactions to make f_1 relax

τ_{int} : ?

① duration of a collision $\quad \tau_{col} \simeq \frac{d}{v}$; with d the particle size

$d \sim 3\,\mathring{A} \Rightarrow \tau_{col} \simeq 6.10^{-13}\,s = 0{,}6\,ps.$

But, most of the time, there are no collisions!

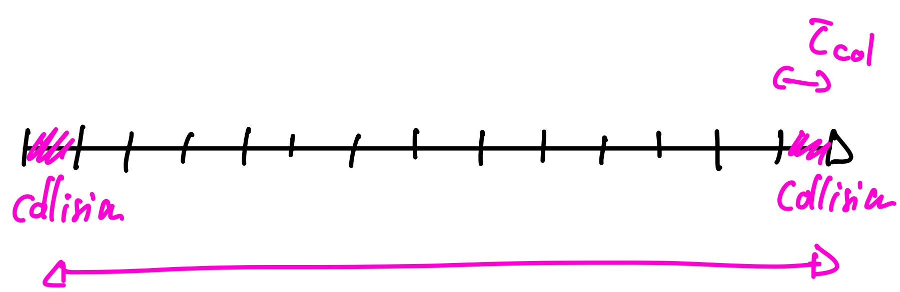
② time between collisions τ_{int}

Every τ_{col}, the particle travels its size. If there is a particle ahead, there is a collision, otherwise, there is no collision.

proba of collision: $p_c = d^3 \times n$, where n is the gas number density.

$PV = NkT \Rightarrow n = \frac{N}{V} = \frac{P}{kT} \simeq 2.4 \times 10^{25}\, m^{-3} \Rightarrow p_c = \frac{d^3 P}{kT} \simeq 6.5 \times 10^{-4}$

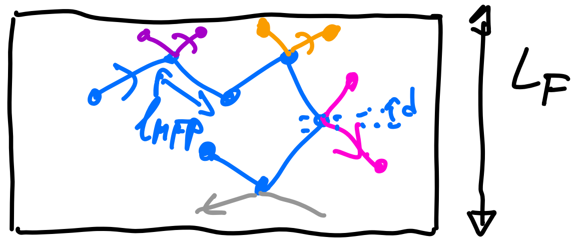
It takes n_t time intervals for a collision to occur where $p n_t \simeq 1$



$\Rightarrow n_t = \frac{kT}{d^3 p} \simeq 1500$

$\tau_{int} = n_t \tau_{col} = \frac{kT}{d^2 P v} \simeq 9 \cdot 10^{-10}\, s$

Mean free path: $\ell_{MFP} = v\,\tau_{int} \sim 460\,\mathrm{nm}$



Three well separated time scales in the system: $\underset{10^{-2}}{\tau_F} \gg \underset{10^{-9}}{\tau_{int}} \gg \underset{10^{-12}}{\tau_{col}}$

$$\tau_F \sim 10^{-2}\,s \gg \tau_{int} \sim 10^{-9}\,s \gg \tau_{col} \sim 10^{-12}\,s$$

$$L_F \sim 1\,m \gg \ell_{MFP} \sim 100\,\mathrm{nm} \gg d \sim 0.5\,\mathrm{nm}$$

Boltzmann equation:

Describe the evolution over τ such that: $\tau_{int} \gg \tau \gg \tau_{col}$

→ collisions look instantaneous

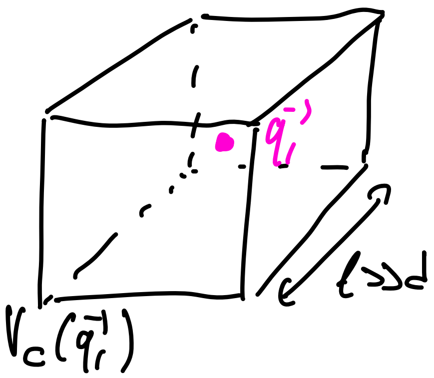
→ ——— are still rare and somewhat random events.

2.2.2) The coarse graining

Start from
$$\partial_t f_1 + \{f_1, H_1\} = \int d\vec{q}_2\, d\vec{p}_2\, \frac{\partial V(\vec{q}_1 - \vec{q}_2)}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} f_2(\vec{q}_1, \vec{p}_1, \vec{q}_2, \vec{p}_2)$$

Build a coarse grained dynamics over scales τ, ℓ such that

$$\tau_{int} \gg t \gg \tau_{col} \quad \text{and} \quad \ell_{MFP} \gg \ell \gg d$$



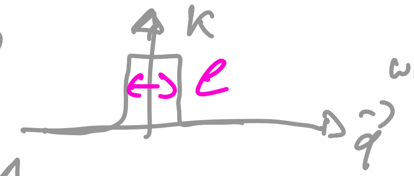
Local number density in phase space

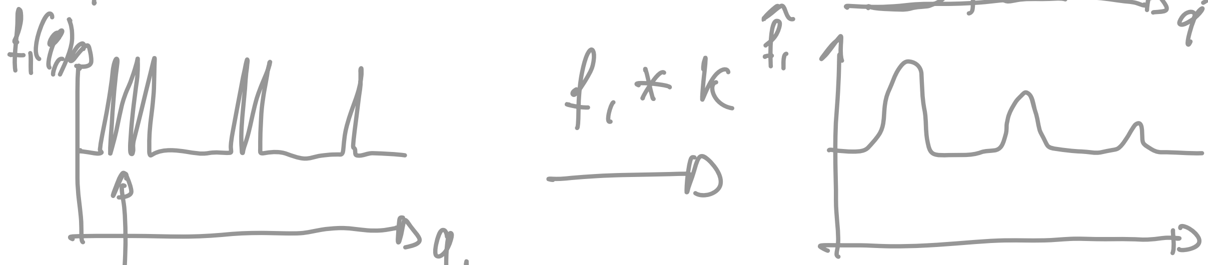
$$\hat{f}_1(\vec{q}_1, \vec{p}_1, t) \simeq \frac{1}{V_c} \int_{V_c(\vec{q}_1)} d\vec{q}_1{}'\, f_1(\vec{q}_1{}', \vec{p}_1, t) \quad (CG)$$

average density within a finite volume near $\vec{q}_1$.

particles "infinitely close to $\vec{q}_1{}'$"

Comment: (CG) can be seen as a convolution between $f_1(\vec{q}_1,\vec{p}_1,t)$ and a filter on $\vec{q}_1$ with $\int k = 1$





δ function due to possible presence of particle

Time evolution of $\hat{f}_1(\vec{q}_1,\vec{p}_1,t)$:

$$\frac{1}{V_c}\int_{V_c} d\vec{q}_1{}'\,\partial_t f_1(\vec{q}_1{}',\vec{p}_1,t) = \partial_t \hat{f}_1(\vec{q}_1,\vec{p}_1,t)$$

$$= \underbrace{-\frac{1}{V_c}\int_{V_c} d\vec{q}_1{}'\,\{f_1(\vec{q}_1{}',\vec{p}_1,t),H_1\}}_{\text{(I), the free evolution}} + \underbrace{\frac{1}{V_c}\int_{V_c} d\vec{q}_1{}'\int d\vec{q}_2\,d\vec{p}_2\,\frac{\partial V(\vec{q}_1{}'-\vec{q}_2)}{\partial\vec{q}_1}\cdot\frac{\partial f_2}{\partial\vec{p}_1}}_{\text{(II) the interaction term } \equiv \left.\frac{\partial\hat{f}_1}{\partial t}\right|_{col}}$$

2.2.2.1) The free evolution

Since nothing happens on scale ℓ due to H_1, we expect that

$$\partial_t \hat{f}_1 + \{\hat{f}_1, H_1\} = [\text{evolution due to collisions}] \equiv \left.\frac{\partial\hat{f}_1}{\partial t}\right|_{col.}$$

Compute (I):

$$(*)\quad \frac{1}{V_c}\int_{V_c} d\vec{q}_1{}'\,\underbrace{\frac{\partial H}{\partial\vec{q}_1}}_{\sim\text{ constant in } V_c}\cdot\frac{\partial}{\partial\vec{p}_1} f_1 \simeq \left.\frac{\partial H}{\partial\vec{q}_1}\right|_{\vec{q}_1}\cdot\frac{\partial}{\partial\vec{p}_1}\underbrace{\frac{1}{V_c}\int d\vec{q}_1{}'\,f_1}_{=\hat{f}_1} = \boxed{\frac{\partial H}{\partial\vec{q}_1}\cdot\frac{\partial\hat{f}_1}{\partial\vec{p}_1}}$$

$$(*)\quad -\frac{1}{V_c}\int d\vec{q}_1{}'\,\frac{\partial H}{\partial\vec{p}_1}\cdot\frac{\partial f_1(\vec{q}_1{}',\vec{p}_1,t)}{\partial\vec{q}_1} = -\frac{\partial H}{\partial\vec{p}_1}\cdot\left[\frac{1}{V_c}\int d\vec{q}_1\,\frac{\partial f_1(\vec{q}_1,\vec{p}_1,t)}{\partial\vec{q}_1}\right]$$

5

By linearity, the spatial average of the gradient is the gradient of the spatial average

$$\int_{V_c(\vec{q}_1)} d\vec{q}_1{}' \frac{\partial f}{\partial \vec{q}_1}(\vec{q}_1{}', \vec{p}_1, t) = \frac{\partial}{\partial \vec{q}_1} \int_{V_c(\vec{q}_1)} d\vec{q}_1 \, f(\vec{q}_1{}', \vec{p}_1, t) = \frac{\partial}{\partial \vec{q}_1} \hat{f}_1(\vec{q}_1, \vec{p}_1, t)$$

$$\Rightarrow -\frac{1}{V_c} \int d\vec{q}_1{}' \frac{\partial H}{\partial \vec{p}_1} \cdot \frac{\partial f_1}{\partial \vec{q}_1}(\vec{q}_1{}', \vec{p}_1, t) = \boxed{-\frac{\partial H}{\partial \vec{p}_1} \cdot \frac{\partial \hat{f}_1}{\partial \vec{q}_1}}$$

Proof: 1d: $F(x) = \frac{1}{2\ell} \int_{x-\ell}^{x+\ell} dy \, f(y)$

$$\Rightarrow F'(x) \overset{\text{Leibniz}}{=} \frac{1}{2\ell} \left(f(x+\ell) - f(x-\ell) \right) = \frac{1}{2\ell} \int_{x-\ell}^{x+\ell} f'(y) dy$$

nd: $F(\vec{x}) = \frac{1}{V} \int_{V(\vec{x})} d^n\vec{y} \, f(\vec{y})$

For any $d\vec{\ell}$, $F(\vec{x}+d\vec{\ell}) - F(\vec{x}) \simeq \vec{\nabla} F \cdot d\vec{\ell}$

$$= \frac{1}{V} \left[\int_{V(\vec{x}+d\vec{\ell})} f(\vec{y}) d^n\vec{y} - \int_{V(\vec{x})} f(\vec{y}{}') d^n\vec{y}{}' \right]$$

$\vec{y} \to \vec{y} + d\vec{\ell}$

$$= \frac{1}{V} \left[\int_{V(\vec{x})} \left[f(\vec{y} + d\vec{\ell}) - f(\vec{y}) \right] d^n\vec{y} \right]$$

$$= \frac{1}{V} \left[\int_{V(\vec{x})} \vec{\nabla} f(\vec{y}) \cdot d\vec{\ell} \; d^n\vec{y} \right]$$

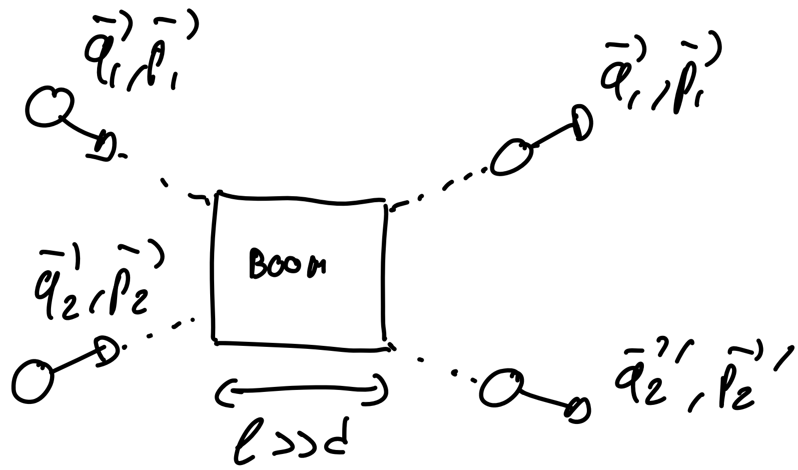
$$= \left[\frac{1}{V} \int_{V(\vec{x})} \vec{\nabla} f(\vec{y}) d\vec{y} \right] d^n\vec{\ell} \Rightarrow \boxed{\vec{\nabla} F = \frac{1}{V} \int_{V(\vec{x})} \vec{\nabla} f(y) d^n\vec{y}}$$

All in all, upon spatial coarse graining,

$$\boxed{\partial_t f_1 + \{f_1, H_1\} \xrightarrow{\frac{1}{V_c}\int d\vec{q}_1} \partial_t \hat{f}_1 + \{\hat{f}_1, H_1\}}$$

This entirely accounts for transport and we now need to account for how the collisions, i.e compute $\left.\frac{\partial \hat{f}_1}{\partial t}\right|_{col}$

2.2.2.2) The collisions



Resolving precisely what happens during collisions is very difficult.

$\Rightarrow$ Can we build directly the resulting dynamics for $\hat{f}_1$?

Time-scale

* On times $\sim \tau_{col}$, very rare collisions, the average phase space density barely evolves
* On time $\sim \tau_{MFP}$, frequent collisions $\Rightarrow \hat{f}_1$ evolves $\Rightarrow \tau_{MFP}$ is a natural evolution time of $\hat{f}_1 \Rightarrow \hat{f}_1 = \hat{f}_1\left(\frac{t}{\tau_{MFP}}\right)$
* Consider $\tau_{MFP} \gg \tau \gg \tau_{col}$, few collisions here and there so that the average phase space density does not evolve much and